

The linearity of quantum mechanics from the perspective of Hamiltonian cellular automata

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Abstract. We discuss the action principle and resulting Hamiltonian equations of motion for a class of integer-valued cellular automata introduced recently [1]. Employing sampling theory, these deterministic finite-difference equations are mapped reversibly on continuum equations describing a set of bandwidth limited harmonic oscillators. They represent the Schrödinger equation. However, modifications reflecting the bandwidth limit are incorporated, *i.e.*, the presence of a time (or length) scale. When this discreteness scale is taken to zero, the usual results are obtained. Thus, the linearity of quantum mechanics can be traced to the postulated action principle of such cellular automata and its conservation laws to discrete ones. The cellular automaton conservation laws are in one-to-one correspondence with those of the related quantum mechanical model, while admissible symmetries are not.

1. Introduction

The linearity of quantum mechanics (QM) is a fundamental feature most notably embodied in the Schrödinger equation. This linearity does not depend on the particular object under study, provided it is sufficiently isolated from anything else. It is naturally reflected in the superposition principle and entails the “quantum essentials” interference and entanglement, without which modern applications of QM, *e.g.* in precision measurement and information technologies, and the ongoing study of the foundations of QM would not be the same.

The linearity of QM has been questioned now and then and various nonlinear modifications have been proposed, in order to test experimentally the robustness of QM against such nonlinear deformations. This has been thoroughly discussed by Jordan presenting a stepwise proof that QM has to be linear based on the separability assumption “... *that the dynamics we are considering can be independent of something else in the universe, that the system we are considering can be described as part of a larger system without interaction with the rest of the larger system*” [2]. It is also worth recalling that most proposals of nonlinearities have been beset with the problem of superluminal signalling or communication between branches of the wave function, see, *e.g.*, Refs. [3, 4, 5, 6]; since then, “no signalling” has become an important criterium that attempted modifications of QM are confronted with.

We have recently provided a different point of view concerning the linearity of QM [1]. We have related QM to the mechanics of a class of Hamiltonian cellular automata. Which shows that the linearity of QM is necessarily required by the consistency of the postulated action principle for such discrete dynamics. This may further the understanding of interference, entanglement, and measurement in QM and lead to new approximation schemes in quantum theory.

Our approach was motivated by explorations of discrete deterministic mechanics by Lee [7], by the study of bandwidth limited fields and their possible role in the physics of gravity and spacetime by Kempf [8], and by the representation of QM in terms of classical notions (observables, phase space, Poisson bracket algebra) by Heslot [9, 10]. Presently, we recover our earlier arguments, which combine these ideas, and add a few details on the way.

2. Discrete Hamiltonian mechanics

Discreteness has many facets in physics, besides quantization, *e.g.*, discrete maps for numerical studies of complex systems, regularized versions of quantum field theories on spacetime lattices, or intrinsically discrete processes. Finite difference equations are expected to play a prominent role here, instead of the usual preponderance of differential calculus.

Lee and collaborators proposed to incorporate fundamental discreteness into all of dynamics [7] (and references therein). This was apparently motivated by the difficulties encountered in trying to formulate a consistent theory of “quantum gravity”, or even “the” unified theory. Thus, deterministic discrete mechanics derives from the assumption that *time is a discrete dynamical variable*. This invokes a *fundamental time or length scale* (in natural units), l , and can be rephrased that in a fixed $(d + 1)$ -dimensional spacetime volume Ω maximally Ω/l^{d+1} measurements can be performed or this number of events take place [7].

We consider the new discreteness scale l in the spirit of *deformations* of Lorentz symmetry, in the form of “doubly special relativity” (DSR) [11], or in its explicit breaking, and in the nonlinear *deformation* of QM [3, 4, 5, 6] mentioned earlier. While such studies necessarily introduce additional parameters, the aim is to probe the stability of the existing theories, the Standard Model and QM in particular, against such deformations, predicting new phenomena that eventually could lead to a deeper theory with a smaller set of fundamental parameters [12]. It is commonly expected that $l \equiv l_{Pl}$, *i.e.*, that discreteness and Planck scale coincide.

Various discrete models have been elaborated, which share desirable symmetries with the corresponding continuum theories while presenting finite degrees of freedom. Different forms and (dis)advantages of a Lagrangian formulation [7] have been discussed, *e.g.*, in Refs. [13, 14, 15]. In the following, we introduce an action principle instead which leads to particularly transparent and symmetric Hamiltonian equations of motion, corresponding to a discrete phase space picture.

3. The action principle for Hamiltonian cellular automata

The state of a classical cellular automaton (CA) with a denumerable set of degrees of freedom will be represented by *integer-valued* “coordinates” x_n^α, τ_n and “conjugated momenta” p_n^α, π_n , where $\alpha \in \mathbf{N}_0$ denote different degrees of freedom and $n \in \mathbf{Z}$ different states.

The x_n and p_n might be higher dimensional vectors, while τ_n and \mathcal{P}_n are assumed one-dimensional. In separating the “coordinate” τ_n from the x_n^α ’s (and correspondingly π_n from the p_n^α ’s), we follow Refs. [7, 13, 14]. This degree of freedom represents the *dynamical time* variable here, instead of the external parameter time of Newtonian mechanics or QM.

Finite differences, for all dynamical variables, are defined by

$$\Delta f_n := f_n - f_{n-1} \quad . \quad (1)$$

Furthermore, we define (using henceforth the summation convention for Greek indices, $r^\alpha s^\alpha \equiv \sum_\alpha r^\alpha s^\alpha$):

$$\mathcal{A}_n := \Delta \tau_n (H_n + H_{n-1}) + a_n \quad , \quad (2)$$

$$H_n := \frac{1}{2} S_{\alpha\beta} (p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta} p_n^\alpha x_n^\beta + R_n \quad , \quad (3)$$

$$a_n := c_n \pi_n \quad , \quad (4)$$

where constants, c_n , and symmetric, $\hat{S} \equiv \{S_{\alpha\beta}\}$, and antisymmetric, $\hat{A} \equiv \{A_{\alpha\beta}\}$, matrices are all integer-valued; R_n stands for higher than second powers in x_n^α or p_n^α . The choice of the right-hand side of Eq. (4) determines the behaviour of the variable τ_n ; for our present purposes, a very simple choice suffices and will be further discussed shortly (nontrivial ‘potential’ terms have been made use of in Ref. [16]).

In terms of the definitions just given, we introduce the integer-valued CA *action*

$$\mathcal{S} := \sum_n [(p_n^\alpha + p_{n-1}^\alpha) \Delta x_n^\alpha + (\pi_n + \pi_{n-1}) \Delta \tau_n - \mathcal{A}_n] , \quad (5)$$

and postulate that a Hamiltonian CA is described by the following

Action Principle: The CA obeys the discrete updating rules (equations of motion) which are determined by $\delta\mathcal{S} = 0$, referring to arbitrary integer-valued variations of all dynamical variables,

$$\delta g(f_n) := [g(f_n + \delta f_n) - g(f_n - \delta f_n)]/2 , \quad (6)$$

where f_n stands for one of the variables on which polynomial g may depend. •

Several remarks are in order here. – We observe that the variations of constant, linear, or quadratic terms yield results that are analogous to the continuum case. – While infinitesimal variations do not conform with integer valuedness, there is no *a priori* constraint on integer ones. However, for arbitrary δf_n , the *remainder of higher powers* in Eq. (3), which enters the action, has to vanish for consistency, $R_n \equiv 0$. Otherwise the number of equations of motion generated by the action principle, generally, would exceed the number of variables. We note that a suitably chosen R_0 or a sufficient small number of such remainder terms could encode the necessary *initial conditions* for the CA evolution.

With the notation $\dot{O}_n := O_{n+1} - O_{n-1}$, the following CA *equations of motion* are obtained:

$$\dot{x}_n^\alpha = \dot{\tau}_n (S_{\alpha\beta} p_n^\beta + A_{\alpha\beta} x_n^\beta) , \quad (7)$$

$$\dot{p}_n^\alpha = -\dot{\tau}_n (S_{\alpha\beta} x_n^\beta - A_{\alpha\beta} p_n^\beta) , \quad (8)$$

$$\dot{\tau}_n = c_n , \quad (9)$$

$$\dot{\pi}_n = \dot{H}_n , \quad (10)$$

which are discrete analogues of Hamilton’s equations, where all terms are integer-valued. Discreteness of the *automaton time* n is reflected in the finite difference equations here.

Note that the $\dot{\tau}_n$ present *background* parameters for the evolving x, p -variables, as a consequence of Eqs. (4), (9). Generally, $\dot{\tau}$ is a *lapse function* in Eqs. (7)–(8).

The Eqs. (7)–(10) are *time reversal invariant*; the state $n+1$ can be calculated from knowledge of the earlier states n and $n-1$ and the state $n-1$ from the later ones $n+1$ and n .

4. Conservation laws and CA symmetries

Surprisingly, there are conservation laws that are always respected by the discrete equations of motion Eqs. (7) and (8). – Introducing the self-adjoint matrix $\hat{H} := \hat{S} + i\hat{A}$, these equations can be combined into:

$$\dot{x}_n^\alpha + i\dot{p}_n^\alpha = -i\dot{\tau}_n H_{\alpha\beta} (x_n^\beta + ip_n^\beta) , \quad (11)$$

and its adjoint. Thus, we recover a *discrete analogue of Schrödinger’s equation*, with $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$ as the amplitude of the “ α -component” of “state vector” $|\psi\rangle$ at “time” n . Then, the Eqs. (7)–(8) imply this:

Theorem A: For any matrix \hat{G} that commutes with \hat{H} , $[\hat{G}, \hat{H}] = 0$, there is a *discrete conservation law*:

$$\psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^\beta = 0 . \quad (12)$$

For self-adjoint \hat{G} , with complex integer elements, this relation concerns real integer quantities. •

Corollary A: For $\hat{G} := \hat{1}$, the Eq. (12) implies a *conserved constraint* on the state variables:

$$\psi_n^{*\alpha} \dot{\psi}_n^\alpha + \dot{\psi}_n^{*\alpha} \psi_n^\alpha = 0 . \quad (13)$$

For $\hat{G} := \hat{H}$, an *energy conservation law* follows. •

Note that Eqs. (12) and (13) *cannot* be trivially “integrated”, since the *Leibniz rule* is modified. Recalling $\dot{O}_n := O_{n+1} - O_{n-1}$, we have, for example, $O_{n+1} O'_{n+1} - O_{n-1} O'_{n-1} = \frac{1}{2}(\dot{O}_n [O'_{n+1} + O'_{n-1}] + [O_{n+1} + O_{n-1}] \dot{O}'_n)$, instead of the product rule of differentiation.

Furthermore, we cannot obtain a continuum limit simply by letting the discreteness scale $l \rightarrow 0$, as for example in Refs. [7, 16]. integer valuedness here conflicts with continuous time and related derivatives.

4.1. Another CA action and conservation laws without admissible unitary symmetries?

The CA action is invariant under suitable unitary transformations. This can be most easily recognized by considering an equivalent form of the action, *i.e.*, which generates the same discrete equations of motion as before. We may replace the definition given in Eq. (5) by:

$$\mathcal{S} := \sum_n [\text{Im}(\psi_n^{\alpha*} \psi_{n-1}^\alpha) + (\pi_n + \pi_{n-1}) \Delta\tau_n - \mathcal{A}_n] , \quad (14)$$

with $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$ and $\text{Im}X := (X - X^*)/2i$, together with the replacement of the definition of H_n (recall $R_n \equiv 0$), Eq. (3), by:

$$H_n := \frac{1}{2} H_{\alpha\beta} \psi_n^{\alpha*} \psi_n^\beta , \quad (15)$$

which enters the action through \mathcal{A}_n , Eq. (2). Then, n -independent unitary transformations \hat{U} , with $\psi'_n = \hat{U} \psi_n$ and $[\hat{U}, \hat{H}] = 0$, leave the action \mathcal{S} invariant.

The self-adjoint matrices \hat{G} of *Theorem A* generate unitary transformations which leave \mathcal{S} invariant. However, since the CA variables $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$ are restricted to be *complex integer-valued*, only unitary transformations that preserve this property are *admissible*. In general, we expect that there are very few interesting ones. Thus, we encounter here the situation that there can exist CA conservation laws, according to *Theorem A* or *Corollary A*, which are *not* related to *admissible symmetry transformations*. This differs from what is usually the case in QM.

5. Sampling theory

It is worth recalling the underlying assumption of discrete mechanics that the density of events and, thus, of information content of spacetime regions is cut off by the scale l [7, 17]. Therefore, despite the observed similarities of our Hamiltonian CA with QM systems, we may wonder whether the discreteness of a deterministic CA can be reconciled with any continuum description at all and, in particular, with QM?

Indeed, we have argued that physical fields, wave functions in particular, could be *simultaneously discrete and continuous*, represented by sufficiently smooth functions containing a finite density of degrees of freedom [1]. This idea has recently been introduced by Kempf and

has led to a covariant ultraviolet cut-off suitable for theories including gravity [8]. However, neither *integer-valued CA* nor the *structure of QM* have been addressed in this way.

The fact that information can have simultaneously continuous and discrete character has been pointed out by Shannon in his pioneering work [18]. This is routinely applied in signal processing, converting analog to digital encoding and *vice versa*. Sampling theory demonstrates that a bandlimited signal can be perfectly reconstructed, provided discrete samples of it are taken at the rate of at least twice the band limit (Nyquist rate). For an extensive review and modern developments of the theory, see Refs. [19, 20], respectively.

We consider the *Sampling Theorem* in its simplest form [8, 19]: Consider square integrable *bandlimited functions* f , *i.e.*, which can be represented as $f(t) = (2\pi)^{-1} \int_{-\omega_{max}}^{\omega_{max}} d\omega e^{-i\omega t} \tilde{f}(\omega)$, with bandwidth ω_{max} . Given the set of amplitudes $\{f(t_n)\}$ for the set $\{t_n\}$ of equidistantly spaced times (spacing π/ω_{max}), the function f is obtained for all t by:

$$f(t) = \sum_n f(t_n) \frac{\sin[\omega_{max}(t - t_n)]}{\omega_{max}(t - t_n)} . \quad (16)$$

Since the CA time is given by the integer n , the corresponding discrete *physical time* is obtained by multiplying with the fundamental scale l , $t_n \equiv nl$, and the bandwidth by $\omega_{max} = \pi/l$.

When attempting to *map invertibly* Eqs. (7)–(8) on reconstructed continuum equations, according to Eq. (16), the nonlinearity on the right-hand sides is problematic: the product of two functions, with bandwidth ω_{max} each, is not a function with the same bandwidth. Therefore, we presently assume that $\dot{\tau}_n$ is a constant and postpone consideration of more general situations.

Let us recall Eq. (11). Inserting $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$ and applying the *Sampling Theorem*, this discrete time equation is mapped to the *continuous time equation*:

$$\frac{\hat{D}_l - \hat{D}_{-l}}{2} \psi^\alpha(t) = \sinh(l\partial_t) \psi^\alpha(t) = \frac{1}{i} H_{\alpha\beta} \psi^\beta(t) , \quad (17)$$

where we employed the translation operator defined by $\hat{D}_T f(t) := f(t + T)$ and set $\dot{\tau}_n \equiv \dot{\tau} = 2$.

Thus, we obtain the *Schrödinger equation*, however, modified in important ways. (We use the QM terminology freely and concentrate on new effects arising here.) The wave function ψ^α has bandwidth ω_{max} , due to reconstruction formula (16). This leads to an *ultraviolet cut-off* of the energy E of stationary states of the generic form $\psi_E(t) := \exp(-iEt) \tilde{\psi}$. Diagonalizing the self-adjoint Hamiltonian, $\hat{H} \rightarrow \text{diag}(\epsilon_0, \epsilon_1, \dots)$, Eq. (17) yields the eigenvalue equation:

$$\sin(E_\alpha l) = \epsilon_\alpha , \quad (18)$$

and a *modified dispersion relation*, $E_\alpha = l^{-1} \arcsin(\epsilon_\alpha) = l^{-1} \epsilon_\alpha [1 + \epsilon_\alpha^2/3! + \mathcal{O}(\epsilon_\alpha^4)]$ [21]. The spectrum $\{E_\alpha\}$ is cut off by the condition $|\epsilon_\alpha| \leq 1$, entailing $|E_\alpha| \leq \pi/2l = \omega_{max}/2$, *i.e.* half the bandlimit.

For CA with a finite (or truncated) number of degrees of freedom or states, labeled by α , the constant $\dot{\tau}$ could be determined instead by the largest eigenvalue of H , $\epsilon_{max} := \max\{|\epsilon_\alpha|\}$, choosing $\dot{\tau} = 1/\epsilon_{max}$. In this way, there would be stationary states corresponding to all eigenvectors of \hat{H} . Otherwise, states corresponding to larger eigenvalues of \hat{H} are unstable, related to complex solutions of the dispersion relation.

The modified Schrödinger equation (17) incorporates an infinite series of higher-order time derivatives. These are negligible for low-energy wave functions, which vary little with respect to the cut-off scale, *i.e.* $|\partial^k \psi / \partial t^k| \ll l^{-k} = (\omega_{max}/\pi)^k$. However, in general, they would require an infinity of initial data.

5.1. Continuous time conservation laws

The relation between Eq. (11) and Eq. (17), together with the linearity of both equations, suggest that the correct continuous time conservation laws can be obtained by the replacement

$$\dot{\psi}_n := \psi_{n+1} - \psi_{n-1} \longrightarrow \frac{1}{i} \sin(il\partial_t)\psi(t) , \quad (19)$$

from Eqs. (12) and (13), respectively. Indeed, by Eq. (17), the following holds:

Theorem B: For any matrix \hat{G} with $[\hat{G}, \hat{H}] = 0$, there is a *continuous time conservation law*:

$$\psi^{*\alpha} G_{\alpha\beta} \sin(il\partial_t)\psi^\beta + [\sin(il\partial_t)\psi^{*\alpha}] G_{\alpha\beta} \psi^\beta = 0 , \quad (20)$$

in particular,

$$\psi^{*\alpha} \sin(il\partial_t)\psi^\alpha + [\sin(il\partial_t)\psi^{*\alpha}] \psi^\alpha = 0 , \quad (21)$$

which modifies the QM wave function *normalization*, referring to a basis labeled by α . •

Only now we can remove the ultraviolet cut-off, with $l \rightarrow 0$, and recover familiar QM results from the leading order terms in Eqs. (20)–(21). (If l is a fundamental *constant*, this limit may be interesting for heuristic reasons alone.)

For example, consider the real symmetric *two-time function*,

$$2C_{\hat{G}}(t_1, t_2) := \psi^{*\alpha}(t_1) G_{\alpha\beta} \psi^\beta(t_2) + \text{c.c.} , \quad (22)$$

where $X + \text{c.c.} := X + X^*$ and \hat{G} is a self-adjoint matrix, with $[\hat{G}, \hat{H}] = 0$. Applying *Theorem B*, we obtain:

Corollary B: The two-time function $C_{\hat{G}}$ is invariant under discrete translations of this form:

$$C_{\hat{G}}(t - l, t) = C_{\hat{G}}(t, t + l) , \quad (23)$$

implying that it is fixed everywhere by giving $C_{\hat{G}}(t, t + l)$ for all t in an interval $[t_0, t_0 + l]$. •

The wave function normalization, $\psi^{*\alpha}\psi^\alpha = 1$, then arises here from the coincidence limit of a two-time function with the property $C_1(t, t + l) \equiv 1$, for all t :

$$1 = \lim_{l \rightarrow 0} C_1(t, t + l) = \psi^{*\alpha}(t)\psi^\alpha(t) , \quad (24)$$

which is consistent with Eq. (21) and essential for the probability interpretation in QM. An analogous *equal-time* constraint, in general, does not exist on the CA level of description. *E.g.*, $\psi_n^{*\alpha}\psi_n^\alpha = x_n^\alpha x_n^\alpha + p_n^\alpha p_n^\alpha = 1$, instead of Eq. (13), is compatible only with rather trivial evolution, since all variables are integer-valued.

It is remarkable how properties of Hamiltonian CA produce familiar QM results, even if modified by the finite scale l . The operators or matrices that generate the QM conservation laws do so for the bandwidth limited continuum theory as well, as stated by *Theorem B*. Since the *same* vanishing commutator is responsible for the CA conservation laws, Eqs. (12)–(13), they correspond to each other one-to-one. Yet the QM symmetry transformations, generally, comprise a larger set than the admissible discrete ones for CA, which have to respect complex integer valuedness of the dynamical variables.

These observations leave us with an intriguing question: *What, if any, would be physical reasons for the existence of Hamiltonian CA conservation laws that are not tied to symmetries, which are fully developed only in the continuum limit?*

6. Discussion

It will be important to extend the CA–QM map to relativistic QM and QFT. Since wave equations and functional Schrödinger equation are linear and have a Hamiltonian formulation, it should be possible to employ a generalized *Sampling Theorem* for fields. It has been shown indeed that the d’Alembert operator can be covariantly regularized by imposing a finite bandwidth of its spectrum [8], a useful ingredient. – Conversely, given a *Hamiltonian* CA, one could invoke the path integral for classical systems [22] plus reconstruction formulae, attempting to obtain a relativistic bandwidth limited quantum (field) theory. – Other constructions of CA for *relativistic models* have appeared, which either incorporate QM features from the outset, *e.g.*, for the Dirac equation [23], or derive them, *e.g.*, for bosonic QFT and a superstring model [24]; see also references there. All these models have been *noninteracting*.

Lack of interactions there might be dictated by additional restrictions, such as locality, when placing a CA, say at Planck scale, into physical spacetime as experienced at the scales where QM is tested. It is remarkable that arbitrary QM N -level systems can be described by $2N - 1$ nonrelativistic coupled oscillators in one fictitious space dimension [25]. Could it be that fundamental CA exist in an abstract space and QM and spacetime emerge together from there?

The nonrelativistic Hamiltonian CA *do incorporate interactions* through the matrix elements $H_{\alpha\beta}$. Their x^α, p^α - variables can be embedded into two-dimensional phase space, similarly as in Ref. [25]. Yet other interpretations are possible, such as α labelling sites of a d -dimensional lattice or elements of the Hilbert space arrived at in the QM description. This freedom appears in the nonrelativistic situation studied here without reference to gravitation or dynamical spacetime.

Interactions are also incorporated in a statistical theory of certain matrix models, which leads to QM behaviour emerging from a Gibbs distribution [26]. Similarly as in Refs. [24], this assumes a very particular form of dynamics and it remains to be seen whether gauge theories as, for example, in the Standard Model can be recovered. – In distinction, we do not make any assumptions about particular interactions but explore the mapping between structural features of Hamiltonian CA and of QM. It will be challenging to identify principles that distinguish a physically relevant Hamiltonian and related conservation laws within an “ontology” of CA.

In Ref. [1], we have discussed how the essential features of *entanglement* in QM and *apparent nonlocality* [27] come into play in our approach. It will be most interesting to reconsider questions of entanglement and locality in a relativistic generalization of the present theory. – Observables, measurements, and Born rule can be discussed in the bandwidth limited theory with help of Heslot’s work [9] and implications for CA *per se* deserve further study.

We remark that Hamiltonian CA might be useful to simulate complex QM systems by the mapping on computer friendly *integer-valued* dynamical variables. A few hints in this direction have been mentioned in Ref. [1].

Let us also draw attention to the differences between the presently introduced Hamiltonian CA and quantum cellular or *quantum lattice-gas automata* (QLGA). The QLGA have recently attracted attention, since they are, by construction, discretizations of the Schrödinger equation [28, 29], cf. also [23]. They are specifically made to reproduce a quadratic kinetic energy term in the Schrödinger equation in the continuum limit in configuration space. This involves judicious choice of transformation matrices, i.e. implicitly of a number of dimensionless parameters [29]. However, the Hilbert space structure of QM state space with complex wave functions and linearity and unitarity of their evolution have to be incorporated *ab initio*.

In these respects, our approach differs remarkably: It is based on integer-valued dynamical variables and an underlying action principle. This *implies* linearity and unitarity together with all conservation laws. Unlike the case of QLGA and in accordance with the discussion in Sect. II. of the role of the discreteness scale l , the Hamiltonian CA here provide a *discrete deformation* of QM that reduces to it for $l \rightarrow 0$.

7. Conclusion

A map between Hamiltonian cellular automata (CA) and quantum mechanics (QM) has been constructed by combining elements of *discrete mechanics* [7], *sampling theory* [8], and *Hamiltonian formulation* of QM [9]. Thus, structural features of QM, the Schrödinger picture of evolution and conservation laws in particular, can be seen to originate in integer-valued CA incorporating a fundamental scale.

The postulated action principle refers to arbitrary integer variations of the dynamical CA variables. This enforces the *linearity* of the theory, as we have demonstrated [1].

Consequently, the separability assumption mentioned in Sect.I., which underlies the derivation of the linearity of QM *within* its formal framework, can here be substituted as follows: “... *the dynamics we are considering can be independent of something else in the universe*” [2], if and only if *the CA action is stationary under arbitrary integer variations*.

This provides a new view of linearity and the superposition principle in quantum mechanics with manifold consequences and generalizations to be explored.

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